STUDY OF COMMON FIXED POINT OF WEAKLY COMPATIBLE MAPS IN G-METRIC SPACE

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Abstract:-In this article we study weakly compatible maps and E.A. property of pair of self maps. We have proved common fixed point theorem of weakly compatible mappings in G- Metric space.

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1. Introduction

The fixed point Theory has several applications in various fields such as linear inequalities, Parameterize estimation problems. The Banach Contraction Principle was obtained by S. Banach [1] in 1922.In 1976 G. Jungck [2] proved first common fixed point theorem for commuting maps in usual metric space . The concept of weak commutative maps which is a weaker type of commuting pair of maps was obtained by Sesa [3] and proved some common fixed point results in metric space.

In 1986 Jungck [4] stated the concept of compatible mappings to generalize the concept of weak commutative pair of self maps .Then in 1986 Jungck [5] defined the concept of weakly compatible maps in Metric Space and proved some common fixed point theorems. In 2002, M. Aamri and D.E Moutawakil [6] defined the concept of E.A. Property for pair of self maps. In 1960 Gahler [7] derived a new metric space structure called as 2-Metric Space and claimed that this is more generalized structure of Metric Space. But Some Author Proved that there is no relation between these two metric structures. Ha. etc. all [8] proved that 2-metric need not be continuous of its variables, but usual metric is continuous of its variables.

In 1992 B.C. Dhage [9] introduced new generalized notion of metric space called as D-Metric Space .Mustafa Z. and Sims in 2003 [10] proved some of the results in D-metric Space are invalid.The concept of G-metric space was stated by Mustafa and Sims [11] and proved some results of fixed point in G-metric Space. In 2012 Zead Mustafa [12] proved some theorems of common fixed points for weakly compatible mappings.

2. Preliminaries

Definition 2.1 [11]. Let X be a non empty set and $G: X \times X \times X \to R^{+}$ which satisfies the following axioms

(1) $G(a,b,c)=0$ if $a=b=c$ i.e. for every a,b,c in X coincides.
(2) $G(a,a,b)>0$ for every $a,b,c \in X$ s.t. $a \neq b$.
(3)$G(a,a,b) \leq G(a,b,c)$ $\forall$ $a,b,c \in X$
(4)$G(a,b,c)=G(b,a,c)=G(c,b,a)=$.......(Symmetry in all three variables)
(5)$G(a,b,c) \leq G(a,x,x)+G(x,b,c)$, for all a,b,c,x in X (rectangle inequality)

Then the function G is said to be a generalized Metric Space or G-Metric on X and the pair (X,G) is called G-Metric Space.

Example 2.1 Let $G: X^3 \rightarrow R^+$ s.t. $G(a,b,c) =$ perimeter of the triangle with vertices at a,b,c in R^2, also by taking p in interior of the triangle then rectangle inequality is satisfied and the function G is **Remark** a function on X.

2.1 G-Metric Space is the generalization of the ordinary metric Space that is every G-metric space (X,G) gives ordinary metric space (X, d_G)

$$d_G(a,b) = G(a,b,b) + G(a,a,b)$$

Example 2.2 Let (X,d) be the usual Metric space . Then the function $G: X^3 \rightarrow R^+$ s.t. $G(a,b,c) =$ $\max\{d(a,b), d(b,c), d(a,c)\}$ for all a,b,c in X is a G-Metric space.

Definition 2.2 A G-Metric space (X, G) is said to be symmetric if $G(a,b,b) = G(a,a,b)$ for all $a,b,c \in X$ and if $G(a,b,b) \neq G(a,a,b)$, then G is said to be non symmetric G-Metric space.

Example 2.3 Let X={x,y} and $G: X^3 \rightarrow R^+$defined by $G(x,x,x) = G(y,y,y) = 0, G(x,x,y) = 1,$ $G(x,y,y) = 2$ and extend G to all of X^3 by symmetry in the variables. Then X is a G-Metric space but it is non symmetric since G(x,x,y) $\neq$ G(x,y,y).

Definition 2.3 Let (X,G) be a G-Metric space , let $\{a_n\}$ be a sequence of elements in X. The sequence $\{a_n\}$ is said to be G-convergent to a if $\lim_{m,n\rightarrow\infty} G(a, a_n, a_m) = 0$ i.e. for every $\epsilon > 0$, there is N s.t. $G(a, a_n, a_m) < \epsilon$ for all m,n $\geq N$. It is denoted as $\lim_{n\rightarrow\infty} a_n = a$.

Proposition 2.1 ([11]) If (X,G) be a G-Metric space. Then following are equivalent.

(i) $\{a_n\}$ is G-convergent to a.

(ii) $G(a_n, a_n, a) \rightarrow \infty$ as $n \rightarrow \infty$

(iii) $G(a_n, a, a) \rightarrow \infty$ as $n \rightarrow \infty$

(iv) $G(a_m, a_n, a) \rightarrow \infty$ as $m, n \rightarrow \infty$

Definition 2.4 Let (X,G) be a G-Metric space. A sequence $\{a_n\}$ is called G-Cauchy if , for $\epsilon > 0$ there is an $N \in I^+$ (set of positive Integers) s.t.

$$G(a_n, a_m, a_l) < \epsilon \text{ for all } n,m,l \geq N$$

Proposition 2.2 Let (X,G) be a G-Metric space then the function G(a,b,c) is jointly continious in all three of its variables.

Proposition 2.3 ([11]) Let (X,G) be a G-Metric Space. Then for any a,b,c,x in X, it gives that

(i) If $G(a,b,c) = 0$ then $a = b = c$

(ii) $G(a,b,c) \leq G(a,a,b) + G(a,a,c)$

(iii) $G(a,b,b) \leq 2G(b,a,a)$

(iv) $G(a,b,c) \leq G(a,x,c) + G(x,b,c)$

(v) $G(a,b,c) \leq \frac{2}{3}(G(a,x,x) + G(b,x,x) + G(c,x,x))$

Definition 2.5 If S and T be self maps of a set X .If w=Sx=Tx for some x in X, then x is called coincidence point of S and T.

Definition 2.6 [5] Self maps S and T are said to be weakly compatible if they commute at their conicidence point i.e. if Sx=Tx for some x in X then $STx = TSx$

Example 2.4 Let X=[1, +∞) and G(a,b,c)=|a-b|+|b-c|+|a-c|.

Define S,T : X → X by $S(a) = 2a - 1$ and T(a)=a^2 , a ò X , we say that a=1 is the only coincidence point and S(T(1))=S(1)=1 and

T(S(1))=T(1)=1 , so S and T are weakly compatible.

Definition 2.7 [6] Let S and T be any two self maps on metric space (X,d). The pair of maps S and T are said to satisfy E.A. property if there exists a sequence $\{a_n\}$ in X s.t.

$\lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Ta_n = z$, for some z in X.

Example 2.5 Let X=[-1,1] and let G be the G-metric on X^3 to R^+ defined as follows

G(a,b,c)=|a-b|+|b-c|+|a-c|. Then (X,G) be a G-Metric Space.Let us define Sx=x and Tx= $\frac{x}{4}$.

Then for a sequence $a_n = \frac{1}{n}$. Then this gives $\lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Ta_n = 0$,for 0 in X.

Here the pair of self maps satisfy E.A. property.

3 Main Result

Now we prove common fixed point theorem for the pair of weakly compatible maps for the new contraction.

Theorem 3.1:-Let (X,G) be a G-Metric Space which is Complete. If S and T be Weakly Compatible maps on X into itself, s.t.

(1) $S(X) \subseteq T(X)$

(2) $G(Sa,Sb,Sc) \leq \alpha G(Sa,Tb,Tc) + \beta G(Ta,Sb,Tc) + \gamma G(Ta,Tb,Sc) +$
$\delta G(Sa,Tb,Tc),$ *for* all a,b,c in X & α,β,γ and $\delta \geq 0$

s.t. $0 \leq \alpha + 3\beta + 3\gamma + \delta < 1$

(3)*Subspace* S(X) or T(X) is Complete. Then there exists a Unique Common fixed point of S and T in X.

Proof:-Let us choose a_0 be an any element in X . Since $S(X) \subseteq T(X)$, we construct a sequence $\{b_n\}$ in X s.t. for any a_1 in X , $Sa_0 = Ta_1$ .In general for a_{n+1} s.t.

$b_n = Sa_n = Ta_{n+1}$ for n=0,1,2...... From inequality (2) in hypothesis, we have

$$G(Sa_n,Sa_{n+1},Sa_{n+1}) \leq \alpha G(Sa_n,Ta_{n+1},Ta_{n+1}) + \beta G(Ta_n,Sa_{n+1},Ta_{n+1})$$
$$+\gamma G(Ta_n,Ta_{n+1},Sa_{n+1}) + \delta G(Sa_n,Ta_{n+1},Ta_{n+1})$$

∴ from the above sequence ,we have

$$G(Sa_n,Sa_{n+1},Sa_{n+1}) \leq \beta G(Sa_{n-1},Sa_{n+1},Sa_n) + \gamma G(Sa_{n-1},Sa_n,Sa_{n+1})$$

$$(\because \alpha G(Sa_n,Sa_n,Sa_n) = 0 = \delta G(Sa_n,Sa_n,Sa_n))$$

∵ By symmetry, we have

$$G(Sa_{n-1}, Sa_{n+1}, Sa_n) = G(Sa_{n-1}, Sa_n, Sa_{n+1})$$

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le (\beta+\gamma)G(Sa_{n-1}, Sa_n, Sa_{n+1}) \qquad (1.1)$$

By using rectangular inequality of G- metric space .We have

$$G(Sa_{n-1}, Sa_n, Sa_{n+1}) \le G(Sa_{n-1}, Sa_n, Sa_n) + G(Sa_n, Sa_{n+1}, Sa_n)$$
$$\le G(Sa_{n-1}, Sa_n, Sa_n) + 2G(Sa_n, Sa_{n+1}, Sa_{n+1})$$

(∵ By using Proposition 2.1) from given hypothesis (ii), we have

$$(1-2\beta-2\gamma)G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le (\beta+\gamma)G(Sa_{n-1}, Sa_n, Sa_n)$$

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le \frac{\beta+\gamma}{1-2\beta-2\gamma} G(Sa_{n-1}, Sa_n, Sa_n)$$

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le q_1 G(Sa_{n-1}, Sa_n, Sa_n)$$

Where $q_1 = \dfrac{\beta+\gamma}{1-2\beta-2\gamma} < 1$

By continuing in this way , We get

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \le q_1{}^n G(Sa_0, Sa_1, Sa_1) \qquad (1.2)$$

For all n,m ò I^+ , Let m> n and by using rectangle inequality

Consider

$$G(b_n, b_m, b_m) \le G(b_n, b_{n+1}, b_{n+1}) + G(b_{n+1}, b_{n+2}, b_{n+2})$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots + G(b_{m-1}, b_m, b_m)$$

$$G(b_n, b_m, b_m) \le (q_1^n + q_1^{n+1} + \ldots\ldots\ldots\ldots + q_1^{m-1})G(b_0, b_1, b_1)$$

(∴ by using (2))

$$\le \frac{q_1^n}{1-q_1} G(b_0, b_1, b_1)$$

As $n, m \to \infty$ ∴ R.H.S. of above inequality tends to 0. We have $lim_{n\to\infty} G(b_n, b_m, b_m) = 0$ ∴ The sequence $\{b_n\}$ is a G-Cauchy sequence in X . Since $S(X)$ or $T(X)$ is Complete subspace of X then subsequence of $\{b_n\}$ must get a limit in $T(X)$.

∴ The Sequence $\{b_n\}$ also convergent .Since $\{b_n\}$ Contains a Convergent subsequence in $T(X)$. Say it c_1 . Let $u = Tc^{-1}$ then Tu=c_1 Now we prove that Su=c_1

On putting $a = u, b = a_n$ and c=a_n in (ii), We have

$$G(Su,Sa_n,Sa_n) \le \alpha G(Su,Ta_n,Ta_n) + \beta G(Tu,Sa_n,Ta_n) + \gamma G(Tu,Ta_n,Sa_n) + \delta G(Su,Ta_n,Ta_n)$$

as $n \to \infty$, above inequality becomes

$\beta G(Tu,Sa_n,Ta_n) = \beta G(c_1,c_1,c_1) = 0$ also

$\gamma G(Tu,Ta_n,Sa_n) = G(c_1,c_1,c_1) = 0$

$\therefore$ We have

$G(Su,c_1,c_1) \le \alpha G(Su,c_1,c_1)$

This gives , $Su = c_1$

$\therefore Su = Tu = c_1$ $\therefore$ u is a coincident point of S and T.

As S and T are weakly Compatible $\therefore$ By definition $STu = TSu$ $\therefore Sc_1 = Tc_1$

Now we show that $Sc_1 = c_1$. Suppose $Sc_1 \ne c_1$,

$\therefore G(Sc_1,c_1,c_1) > 0$ In (ii) putting a=$c_1, b = u, c = u$

$\therefore$ We have

$$\begin{aligned} G(Sc_1,c_1,c_1) &= G(Sc_1,Su,Su) \\ &\le \alpha G(Sc_1,Tu,Tu) + \beta G(Tc_1,Su,Tu) \\ &+\gamma G(Tc_1,Tu,Su) + \delta G(Sc_1,Tu,Tu) \\ &=(\alpha + \beta + \gamma + \delta)G(Sc_1,c_1,c_1) \\ &< G(Sc_1,c_1,c_1) \end{aligned}$$

Which is a contradiction. $\therefore$ this gives $Sc_1 = c_1$

$\therefore Sc_1 = Tc_1 = c_1$ $\therefore c_1$ is a Common fixed point of S and T.

To prove Uniqueness,

Suppose that c' is another Common fixed Point of S and T which is distinct from c_1. i.e. $c_1 \ne c'$.

Consider,

$$\begin{aligned} G(c_1,c',c') &= G(Sc_1,Sc',Sc') \\ &\le \alpha G(Sc_1,Tc',Tc') + \beta G(Tc_1,Sc',Tc') \\ &+\gamma G(Tc_1,Tc',Sc') + \delta G(Sc_1,Tc',Tc') \\ &=(\alpha + \beta + \gamma + \delta)G(c_1,c',c') \\ &< G(c_1,c',c') \end{aligned}$$

$$\therefore c_1 = c'$$

Hence proof.

Theorem 3.2 :-If S and T be two maps on a G-metric Space (X,G) into itself which Satisfy

$$(i)\ G(Sa,Sb,Sc) \leq \alpha G(Sa,Tb,Tc) + \beta G(Ta,Sb,Tc) + \gamma G(Ta,Tb,Sc) + \delta G(Sa,Tb,Tc), \textit{for}\text{ all a,b,c in X } \& \ \alpha,\beta,\gamma \text{ and } \delta \geq 0$$
$$\text{s.t. } 0 \leq \alpha + 3\beta + 3\gamma + \delta < 1$$

(ii) T(X) is closed subspace of X.

(iii) S and T satisfies E.A. property. Moreover, If S and T are weakly Compatible self Maps. Then S and T have Unique Common fixed Point in X.

Proof:- Given that S and T satisfies E.A. Property $\therefore$ By definition , there exists a sequence $\{a_n\}$ ò X s.t. $\lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Ta_n = z$ò$X$ Also by (ii) \$T(X)\$ is closed, $\therefore$ every Convergent Sequence of Points of \$T(X)\$ contains limit points.

z ò T (X) $\therefore$ for some y ò X , z=Ty

$\therefore$ from (i) we have

$$G(Sy,Sa_n,Sa_n) \leq \alpha G(Sy,Ta_n,Ta_n) + \beta G(Ty,Sa_n,Ta_n) + \gamma G(Ty,Ta_n,Sa_n) + \delta G(Sy,Ta_n,Ta_n)$$

As n $\to \infty$ and by $0 \leq \alpha + 3\beta + 3\gamma + \delta < 1$,

Consider,

$$G(Sy,z,z) \leq \alpha G(Sy,z,z) + \beta G(z,z,z) + \gamma G(z,z,z) + \delta G(Sy,z,z)$$
$$= (\alpha + \delta) G(Sy,z,z)$$
$$\text{but } (\alpha + \delta) < 1$$
$$\therefore G(Sy,z,z) = 0$$
$$\therefore Sy = z$$
$$\therefore Sy = Ty = z \text{ ò X}$$

$\therefore$ y is the Coincident point of S and T.

Also Given that S and T are weakly Compatible.

$\therefore$ Sz=STy=TSy=Tz

$\therefore Sz = Tz$

Using (i) , We have

$$G(Sz,Sy,Sy) \le \alpha G(Sz,Ty,Ty) + \beta G(Tz,Sy,Ty) + \gamma G(Tz,Ty,Sy) + \delta G(Sz,Ty,Ty)$$

$$\therefore G(Sz,z,z) \le \alpha G(Sz,z,z) + \beta G(Sz,z,z) + \gamma G(Sz,z,z) + \delta G(Sz,z,z)$$

$$\le (\alpha+\beta+\gamma+\delta) G(Sz,z,z)$$

$$\therefore G(Sz,z,z) = 0$$

$$\therefore Sz = z$$

$$\therefore Sz = Tz = z$$

$\therefore$ \$z\$ is a Common Fixed point of S and T.

Conclusion: - Thus we have proved Common fixed theorem for pair of weakly compatible mappings and second result for weakly compatible maps which satisfy E.A. property.

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