Application of Fixed Point Theorem in G-Metric Space for Existence of Solution of Integral Equation.

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Abstract

In this paper we obtain result of existence of solution of Integral equation.We prove this result using generalization of Banach contraction principle in G-Metric space.This is a very easy application of fixed point theorem in G-metric space for beginners.This existence of solution of Integral equation requires less assumptions.

Keywords: G-Metric Space,G-Cauchy sequence,G-convergent Sequence, altering distance function.

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1 Introduction

We know by Banach Fixed point principle [1] which is classical and powerful tool in nonlinear analysis.In 1984 M.S.Khan et. al. [2] generalized Banach Contraction principle.Also B.E.Rhoades [3]proved fixed point theorem by using altering distance function.Later in 2008 P.N.Dutta at.al. [4] generalized Banach contraction principle.Tomanari Suzuki [5] proved fixed point theorem which is generalization of Banach fixed point theorem . Banach Fixed point result is generalized in various directions reader can [6, 7]. Selma Gulyaz et.al. [8] obtained important result of fixed points of cyclic weak contraction and obtained existence and uniqueness of solution of integral equation using fixed point result.Many authors derived existence and uniqueness of solution of integral equation via fixed point theorem,readers can see [9, 10, 11, 12]

2 Preliminaries

In 2006 Z.Mustafa [13] introduced a new structure of metric space called as G-Metric Space. Which is defined as follows.

Definition 2.1. *Let X be a non empty set and* $G : X^3 \to R^+$ *which satisfies the following conditions*

1. $G(a,b,c) = 0\ if a = b = c$ *i.e. every* a, b, c *in* X *coincides.*
2. $G(a,a,b) > 0\ for\ every\ a, b, c\ \epsilon X$ *s.t.* $a \neq b$
3. $G(a,a,b) \leq G(a,b,c),\ \forall a,b,c\ \epsilon\ \ X$ *s.t.* $c \neq b$
4. $G(a,b,c) = G(b,a,c) = G(c,b,a) = \ldots\ldots\ldots$
   (symmetrical in all three variables)
5. $G(a,b,c) \leq G(a,x,x) + G(x,b,c)$, *for all a,b,c,x in X*
   (rectangle inequality)

   Then the function G is said to be generalized metric or simply G-metric on X and the pair (X,G) is said to be G-metric space.

Example 2.2. *Let* $G : X^3 \to R^+$ *s.t.* $G(a,b,c) =$ *perimeter of the triangle with vertices at a,b,c in* R^2 *, also by taking p in the interior of the triangle then rectangle inequality is satisfied and the function G is a G-metric on X.*

Remark 2.3. *G-metric space is the generalization of the ordinary metric space that is every G-metric space is (X,G) defines ordinary metric space* (X, d_G) *by* d_G *(a,b)=*$G(a,b,b) + G(a,a,b)$

Example 2.4. *Let (X,d) be the usual metric space . Then the function $G : X^3 \to R^+$ defined by*

$$G(a, b, c) = max.\{d(a, b), d(b, c), d(c, a)\}$$

for all a,b,c ϵ X is a G-metric space.

Definition 2.5. *A G-metric space (X,G) is said to be symmetric if G(a,b,b)=G(a,a,b) for all a,bϵX and if G(a,b,b)$\neq$ G(a,a,b) then G is said to be non symmetric G-metric space.*

Example 2.6. *Let X=$\{x, y\}$ and $G : X^3 \to R^+$ defined by G(x,x,x)=G(y,y,y)=0, G(x,x,y)=1,G(x,y,y)=2 and extend G to all of X^3 by symmetry in the variables. Then X is a G-metric space but It is non symmetric. since G(x,x,y) $\neq$ G(x,y,y)*

Definition 2.7. *Let (X,G) be a G-metric space, Let $\{ a_n \}$ be a sequence of elements in X .The sequence $\{ a_n \}$ is said to be G-convergent to a if*

$$lim_{m,n\to\infty} G(a, a_n, a_m) = 0$$

i.e for every $\epsilon > 0$ there is N s.t. $G(a, a_n, a_m) < \epsilon$ for all m,n $\geq N$ It is denoted as $a_n \to a$ or $lim_{n\to\infty} a_n = a$

Proposition 2.8. *If (X,G) be a G-metric space. Then the following are equivalent*

1. $\{a_n\}$ *is G-convergent to a.*
2. $G(a_n, a_n, a) \to 0$ *as* $n \to \infty$
3. $G(a_n, a, a) \to 0$ *as* $n \to \infty$
4. $G(a_m, a_n, a) \to 0$ *as* $m, n \to \infty$

Definition 2.9. *Let (X,G) be a G-metric space a sequence $\{a_n\}$ is called G-Cauchy if , for each $\epsilon > 0$ there is an N ϵI^+ (set of positive integers) s.t.*

$$G(a_n, a_m, a_l) < \epsilon \text{ for all } n, m, l \geq N$$

In 1922 S.Banach [?] proved a very important result of fixed points which gives a sufficient condition for existence and uniqueness of fixed point of self map in a complete metric space which states that,

Theorem 2.10. ***Banach Contraction Mapping Principle***
Any Contraction mapping S *of a non empty complete metric space* (X,d) *into itself has a unique fixed point.*

Definition 2.11. *[2] A function* $\phi:[0,\infty)\to[0,\infty)$ *is called an altering distance function if the following properties are satisfied.*

1. $\phi(t)=0$ *if and only if* $t=0$
2. ϕ *is continuous and nondecreasing.*

In 1984 Khan et. al. [2] proved following fixed point theorem.

Theorem 2.12. *Let* (X,d) *be a complete metric space and let* $g:X\to X$ *be a self mapping.Suppose that there exists an altering distance functions* ϕ *and a constant* $k\in[0,1)$ *such that*

$$\phi(d(ga,gb))\leq k\phi(d(a,b))$$

. *Then* g *has a unique fixed point in* X *for all* $a,b\in X$

B.E. Rhoades [3] proved following fixed point result which is as follows.

Theorem 2.13. *Let* (X,d) *be a complete metric space and let* $T:X\to X$ *be a self mapping.Suppose that there exists an altering distance function* ϕ *such that*

$$\phi(d(ga,gb))\leq d(a,b)-\phi(d(a,b))$$

, *for all* $a,b\in X$.*Then* g *has a unique fixed point.*

Also Dutta and Chaudhury [4] proved a generalization of this theorem as follows.

Theorem 2.14. *Let (X,d) be a complete metric space and let $T : X \to X$ be a self mapping.Suppose that there exists two altering distance function ϕ_1 and ϕ_2 such that*

$$\phi_1(d(ga,gb)) \leq \phi_1(d(a,b)) - \phi_2(d(a,b))$$

Then g has a unique fixed point.

Z.Mustafa [14] generalized Banach contraction principle in G-metric space.Which is as follows

Theorem 2.15. *If (X,G) be a complete G-Metric space and F be a mapping of X into itself which satisfies the following condition for all $p,q,r \in X$*

$$G(Fp,Fq,Fr) \leq \beta G(p,q,r)$$

for $0 \leq \beta < 1$ .Then there exists a unique fixed point u of F .

Corollary 2.16. *[15] If (X,G) be a complete G-Metric space and let T be a mapping on X into itself.There exists $\beta \in [0,1)$ such that for all $p,q,r \in X$*

$$G(Tp,Tq,Tq) \leq \beta G(p,q,q)$$

Then T has a unique fixed point $u \in X$

Proof. Let p_0 be arbitrary point in X .We define a sequence $\{p_n\}$ by $P_1 = T(p_0)$, $p_2 = T(p_1)$,....., $p_{n+1} = T(p_n)$ .Then $p_n = T^n(p_0)$, for all $n \in N$ .Consider,

$$\begin{aligned} G(p_n,p_{n+1},p_{n+1}) &\leq G(Tp_{n-1},Tp_n,Tp_n) \\ &\leq \beta G(p_{n-1},p_n,p_n) \\ &= \beta G(Tp_{n-2},Tp_{n-1},Tp_{n-1}) \\ &\leq \beta^2 G(p_{n-2},p_{n-1},p_{n-1}) \\ \therefore G(p_n,p_{n+1},p_{n+1}) &\leq \beta^n G(p_0,p_1,p_1) \qquad (2.1) \end{aligned}$$

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for all $n, m \in N$, $m > n$, and using rectangle inequality, we have

$$\begin{aligned} G(p_n, p_m, p_m) &\leq G(p_n, p_{n+1}, p_{n+1}) + G(p_{n+1}, p_{n+2}, p_{n+2}) + G(p_{n+2}, p_{n+3}, p_{n+3}) \\ &+ \;.... + G(p_{m-1}, p_m, p_m) \\ &\leq (\beta^n + \beta^{n+1} + ... + \beta^{m-1}) G(p_0, p_1, p_1) \\ &= \frac{\beta^n(1-\beta^{m-n})}{1-\beta} G(p_0, p_1, p_1) \end{aligned}$$

Taking limit as $m, n \to \infty$ $\lim_{m,n\to\infty} G(p_n, p_m, p_m) = 0$ $\therefore$ $\{p_n\}$ is a G-Cauchy sequence in X .Since (X, G) is a G-Complete metric space.

$\therefore T(p) = T(\lim_{n\to\infty} p_n) = \lim_{n\to\infty} T(p_n) = p_{n+1} = p$. It gives p is a fixed point of T .To prove uniqueness of fixed point.Suppose q is another fixed point of T . $\therefore$ $T(q) = q$ for any $q \in X$.

$$\begin{aligned} G(p, q, q) &= G(Tp, Tq, Tq) \\ &\leq \beta G(p, q, q) \end{aligned}$$

$\therefore G(p, q, q) = 0$ .Hence $p = q$. $\therefore$ T has a unique fixed point. □

3 Application to Integral Equation

Now we prove our main result of existence of solution of following Integral equation using the the corollary (2.16). The existence of solutions of Integral equations has been obtained by various authors see[9, 10, 11, 12].Here we prove an existence of a solution for a non-linear integral equation in G-metrics space using corollary(2.16). Let us consider the integral equation

$$v(t) = \int_0^L K(t, s, v(s))ds + p(t), \ t \in [0, L] \tag{3.1}$$

Where $L > 0$ .Let $X = C([0, L])$ be the set of all continuous functions defined on closed interval $[0, L]$ .Let $G : X^3 \to R^+$ be defined by

$$G(a, b, c) = Sup_{t\in[0,L]}|a(t) - b(t)| + Sup_{t\in[0,L]}|b(t) - c(t)| + Sup_{t\in[0,L]}|c(t) - a(t)|$$

Then the function G is a G-metric on X and also (X, G) is G-complete metric space.

Theorem 3.1. *We suppose following assumptions hold good.*

1. $K : [0, L] \times [0, L] \times R \to R$ *and* $P : R \to R$ *are continuous.*
2. *There exists a continuous function* $G : [0, L] \times [0, L] \to R^+$ *such that*

$$|K(t, s, v) - K(t, s, u)| \leq G(t, s)|v - u|$$

for every $v, u \in R$ *and* $t, s \in [0, L]$.

3. $Sup_{t\in[0,L]} \int_0^L G(t, s)ds \leq q$ *for some* q *in* $[0, 1)$.

Then the integral equation (6.30) has a solution $v \in X$.

Proof. Let the mapping $F : X \to X$ be defined by

$$Fa(t) = \int_0^L K(t, s, a(s))ds + p(t), \ t \in [0, L]$$

for $a, b \in X$, we get

$$\begin{aligned} G(Fa, Fb, Fc) &\leq 2 \sup_{t\in[0,L]} |Fa(t) - Fb(t)| \\ &= 2 \sup_{t\in[0,L]} |\int_0^L (K(t, s, a(s) - K(t, s, b(s)))ds| \\ &\leq 2 \sup_{t\in[0,L]} \int_0^L |(K(t, s, a(s) - K(t, s, b(s)))|ds \\ &\leq 2 \sup_{t\in[0,L]} |a(t) - b(t)| \sup_{t\in[0,L]} \int_0^L G(t, s)ds \ (using(3)) \\ &= G(a, b, b) \sup_{t\in[0,L]} \int_0^L G(t, s)ds \end{aligned}$$

Since,Using (iii),there exists $q \in [0, 1)$ such that

$$\sup_{t\in[0,L]} \int_0^L G(t, s)ds \leq q$$

. $\therefore$ we have

$$G(Fa, Fb, Fb) \leq qG(a, b, b)$$

Thus all the required conditions of corollary (2.16) are satisfied and $\therefore$ there exists a continuous function $v \in X$ of the integral equation (6.30). □

4 Conclusion

In this paper we have proved result of existence of solution of Integral equation.We have proved this result using generalization of Banach contraction principle in G-Metric space.This is a very easy application of fixed point theorem in G-metric space for beginners.This existence of solution of Integral equation requires less assumptions.

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