Common fixed point result in Complete G-metric space.

V. V. Latpate

A.C.S.College, Gangakhed, Dist. Parbhani, Maharashtra, India 431514

e-mail: *vishnu.latpate@yahoo.com*

Abstract

In this paper we study compatible maps in G-Metric space and obtain a common fixed point Result for pair of Compatible maps in G-Metric space which is the generaliation of common fixed point result of pair of self maps in Complete metric space.

Keywords: G-Metric Space,G-Cauchy sequence,G-convergent Sequence,Compatible maps.

2000 Mathematics Subject Classification: 47H10, 54H25

1 Introduction

In 1976, G.Jungck [1] proved a common fixed point theorem for commuting mappings,which generalizes the Banach Contraction principle.Sesa [2] introduced a concept of weakly commuting mappings and proved some fixed point theorems in complete metric space.Commuting maps are weakly commuting.Jungck's [1] common fixed point theorem has been generalized and modified by many authors [3, 4, 5, 6, 7].In 1986 G.Jungck [5] defined the concept of compatibility and proved some common fixed point results.

In 2006 Mustafa and Sims [9] introduced the concept of G-Metric space.In 2012 Manoj Kumar [8] defined the concept of compatible maps in G-Metric space and proved some results of common fixed points of pair of compatible maps.

2 Preliminaries

Definition 2.1. *Let X be a non empty set and $G : X^3 \to R^+$ which satisfies the following conditions*

1. $G(a,b,c) = 0 \; if a = b = c$ *i.e. every* a, b, c *in* X *coincides.*
2. $G(a,a,b) > 0 \; for \; every \; a,b,c \; \epsilon X$ *s.t.* $a \neq b$
3. $G(a,a,b) \leq G(a,b,c), \; \forall a,b,c \; \epsilon \; X$ *s.t.* $c \neq b$
4. $G(a,b,c) = G(b,a,c) = G(c,b,a) = \ldots\ldots\ldots$
   (symmetrical in all three variables)
5. $G(a,b,c) \leq G(a,x,x) + G(x,b,c)$, *for all a,b,c,x in X*
   (rectangle inequality)

   Then the function G is said to be generalized metric or simply G-metric on X and the pair (X,G) is said to be G-metric space.

Example 2.2. *Let* $G : X^3 \to R^+$ *s.t.* $G(a,b,c) =$ *perimeter of the triangle with vertices at a,b,c in* R^2 *, also by taking p in the interior of the triangle then rectangle inequality is satisfied and the function G is a G-metric on X.*

Remark 2.3. *G-metric space is the generalization of the ordinary metric space that is every G-metric space is (X,G) defines ordinary metric space* (X, d_G) *by*

$$d_G(a,b) = G(a,b,b) + G(a,a,b)$$

Example 2.4. *Let (X,d) be the usual metric space . Then the function* $G : X^3 \to R^+$ *defined by*

$$G(a,b,c) = max.\{d(a,b), d(b,c), d(c,a)\}$$

for all a,b,c ϵ *X is a G-metric space.*

Definition 2.5. *A G-metric space (X,G) is said to be symmetric if G(a,b,b)=G(a,a,b) for all a,bϵX and if G(a,b,b)$\neq$ G(a,a,b) then G is said to be non symmetric G-metric space.*

Example 2.6. *Let X=$\{x, y\}$ and G : $X^3 \rightarrow R^+$ defined by G(x,x,x)=G(y,y,y)=0, G(x,x,y)=1, G(x,y,y)=2 and extend G to all of X^3 by symmetry in the variables. Then X is a G-metric space but It is non symmetric. since G(x,x,y) $\neq$ G(x,y,y)*

Definition 2.7. *Let (X,G) be a G-metric space, Let $\{ a_n \}$ be a sequence of elements in X . The sequence $\{ a_n \}$ is said to be G-convergent to a if*

$$lim_{m,n\rightarrow\infty}G(a, a_n, a_m) = 0$$

i.e for every $\epsilon > 0$ there is N s.t. $G(a, a_n, a_m) < \epsilon$ for all m,n $\geq N$ It is denoted as $a_n \rightarrow a$ or $lim_{n\rightarrow\infty}a_n = a$

Proposition 2.8. *If (X,G) be a G-metric space. Then the following are equivalent*

1. *$\{a_n\}$ is G-convergent to a.*
2. *$G(a_n, a_n, a) \rightarrow 0$ as $n \rightarrow \infty$*
3. *$G(a_n, a, a) \rightarrow 0$ as $n \rightarrow \infty$*
4. *$G(a_m, a_n, a) \rightarrow 0$ as $m, n \rightarrow \infty$*

Definition 2.9. *Let (X,G) be a G-metric space a sequence $\{a_n\}$ is called G-Cauchy if , for each $\epsilon > 0$ there is an N ϵI^+ (set of positive integers) s.t.*

$$G(a_n, a_m, a_l) < \epsilon \; for \; all \; n, m, l \geq N$$

Proposition 2.10. *Let (X,G) be a G-metric space then the function G(a,b,c) is jointly continuous in all three of its variables.*

Proposition 2.11. *Let (X,G) be a G-metric space. Then, for any a,b,c,x in X it gives that*

1. *if $G(a, b, c) = 0$ then $a = b = c$*

2. $G(a,b,c) \leq G(a,a,b) + G(a,a,c)$

3. $G(a,b,b) \leq 2G(b,a,a)$

4. $G(a,b,c) \leq G(a,x,c) + G(x,b,c)$

5. $G(a,b,c) \leq \frac{2}{3}(G(a,x,x) + G(b,x,x) + G(c,x,x))$

Definition 2.12. *[5] Let S and T be two self maps on a metric space (X,d).The mappings S and T are said to be compatible if*

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

, whenever $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \ for \ some \ z\epsilon X$

Definition 2.13. *[8] Let S and T be two self mappings on a G-metric space (X,G).Then mappings S and T are said to be compatible if* $\lim_{n\to\infty} G(STx_n, STx_n, TSx_n) = 0$ *, whenever* $\{x_n\}$ *is a sequence in X s.t.* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \ for \ some \ z\epsilon X$ *for some z in X.*

Example 2.14. *Let X=[-1,1] and* $G : X^3 \to R^+$ *be defined as follows*

$$G(a,b,c) = |a-b| + |b-c| + |c-a|$$

for all a,b,c ϵX. *Then (X,G) be a G-metric space.Let us define fa=a and ga=*$\frac{a}{4}$ *Let* $\{a_n\}$ *be the sequence, s.t.* $a_n = \frac{1}{n}$ *and n is a natural number.It is easy to see that the mappings f and g are compatible as* $\lim_{n\to\infty} G(fga_n, gfa_n, gfa_n) = 0$ *here* $a_n = \frac{1}{n}$ *s.t.* $\lim_{n\to\infty} fa_n = \lim_{n\to\infty} ga_n = 0$ *for* $0\epsilon X$

In 1980,J .Madhusudhanrao [10] proved common fixed point theorem for pair of self maps in Complete metric space.

Theorem 2.15. *Let* (X,d) *be a Complete metric space.If* P , Q *be a pair of maps on* X *into itself and if there exists constants* k_1, k_2, k_3, k_4, k_5 *such that* $0 \leq k_j$, *for* $1 \leq j \leq 5$ *and* $k_1 + k_2 + 2k_3 + 2k_4 + k_5 < 1$ *and*

$$d(Px,Qy) \leq k_1 d(x,Px) + k_2 d(y,Qy) + k_3 d(x,Qy) + k_4 d(y,Px) + k_5 d(x,y)$$

,for all x, y in X . Then P, Q have a unique common fixed point in X .

Now we see some preliminary results of common fixed point theorem as follows. Manoj Kumar Generalized following theorem.Which is stated as

Theorem 2.16. *[8] Let(X,G) be complete G-metric space.Let S and T be self mappings on X satisfying following conditions.*

1. *$S(X) \subseteq T(X)$,*
2. *S or T is continuous ,*
3. *$G(Sa, Sb, Sc) \leq \beta G(Ta, Tb, Tc)$ for every a,b,c in X and $0 \leq \beta < 1$ .And if S and T are Compatible then S and T have Unique common fixed points in X.*

Proof. Let us take a_0 be an arbitrary element of X.We define a sequence s.t. for any point a_1 in X, define $Sa_0 = Ta_1$, $Sa_1 = Ta_2$, $Sa_2 = Ta_3$,....,In general for $a_{n+1} \in X$ s.t. $b_n = Sa_n = Ta_{n+1}$ for $n = 0, 1, 2, 3....$ from (3) we get

$$\begin{aligned} G(Sa_n, Sa_{n+1}, Sa_{n+1}) &\leq \beta G(Ta_n, Ta_{n+1}, Ta_{n+1}) \\ &= \beta G(Sa_{n-1}, Sa_n, Sa_n) \end{aligned}$$

By continuing same procedure, we get

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq \beta^n G(Sa_0, Sa_1, Sa_1) \tag{2.1}$$

$\therefore$ for all $n, m \in N$, $m > n$, by using rectangle inequality, we get

$$\begin{aligned} G(b_n, b_m, b_m) &\leq G(b_n, b_{n+1}, b_{n+1}) + G(b_{n+1}, b_{n+2}, b_{n+2}) \\ &+ G(b_{n+2}, b_{n+3}, b_{n+3}) + ... + G(b_{m-1}, b_m, b_m) \\ &\leq (\beta^n + \beta^{n+1} + ... + \beta^{m-1}) G(b_0, b_1, b_1) \\ &\leq \frac{\beta^n}{1-\beta} G(b_0, b_1, b_1) \end{aligned}$$

taking limit as $n, m \to \infty$, we get $lim_{n,m\to\infty}G(b_n, b_m, b_m) = 0$. $\therefore$ this shows that $\{b_n\}$ is a G-Cauchy sequence in X.Since given (X, G) is G-Complete metric space. $\therefore$,there exists a point $x \in X$ s.t. $lim_{n\to\infty}b_n = x$ and $\therefore lim_{n\to\infty}b_n = lim_{n\to\infty}Sa_n = lim_{n\to\infty}Ta_{n+1} = x$. Since the mapping S or T is Continuous.Suppose T is continuous , $\therefore lim_{n\to\infty}TSa_n = Tx$. also given that S and T are compatible. $\therefore lim_{n\to\infty}G(TSa_n, STa_n, STa_n) = 0$ .This gives $lim_{n\to\infty}STa_n = Tx$ From (3) we get

$$G(STa_n, Sa_n, Sa_n) \leq \beta G(TTa_n, Ta_n, Ta_n)$$

taking limit as $n \to \infty$, we get $Tx = x$ Again from (3)we get

$$G(Sa_n, Sx, Sx) \leq \beta G(Ta_n, Tx, Tx)$$

taking limit as $n \to \infty$, we get $Tx = x$ $\therefore$ we get $Tx = Sx = x$ .Hence x is a common fixed point of S and T. For Uniqueness ,If possible suppose let x_1 be another common fixed point of S and T. Then we have, $G(x, x_1, x_1) > 0$ and

$$\begin{aligned} G(x, x_1, x_1) &= G(Sx, Sx_1, Sx_1) \\ &\leq \beta G(Tx, Tx_1, Tx_1) \\ &= \beta G(x, x_1, x_1) \\ &< G(x, x_1, x_1) \end{aligned}$$

which is impossible. $\therefore$ $x = x_1$ .Hence uniqueness follows. □

Example 2.17. *If $X = [-1, 1]$ and G be a G-metric space s.t. $G : X^3 \to R^+$ defined by*

$$G(x_1, y_1, z_1) = (|x_1 - y_1| + |y_1 - z_1| + |z_1 - x_1|)$$

for all $x_1, y_1, z_1 \in X$ .Then X is a G-Metric space.We define $S(x_1) = \frac{x_1}{6}$ and $T(x_1) = \frac{x_1}{2}$. If S is Continuous and $S(X) \subseteq T(X)$.
Here $G(Sx_1, Sy_1, Sz_1) \leq \beta G(Tx_1, Ty_1, Tz_1)$ is true for all $x_1, y_1, z_1 \in X$, $\frac{1}{3} \leq \beta < 1$ and 0 is the common fixed point of S and T which is Unique.

Now,we prove Common fixed point result for the pair of compatible maps in G-Metric space which is the generalization of Theorem 2.15.

3 Main Result

Theorem 3.1. *Let X be a complete G-metric space. $S,T : X \to X$ be two compatible maps on X and which satisfies the following conditions,*
(i) $S(X) \subseteq T(X)$,
(ii)S or T is G-continuous,
(iii) $G(Sa, Sb, Sc) \leq \alpha G(Sa, Tb, Tc) + \beta G(Ta, Sb, Tc)$
$+ \gamma G(Ta, Tb, Sc) + \delta G(Sa, Tb, Tc)$ for every a,b,c in X and $\alpha, \beta, \gamma, \delta \geq 0$ with $0 \leq \alpha + 3\beta + 3\gamma + 3\delta < 1$. Then S and T have unique common fixed point in X.

Proof. Let a_0 be an arbitrary element in X by $S(X) \subseteq T(X)$,we construct a sequence $\{b_n\}$ in X such that for any a_1 in X $Sa_0 = Ta_1$ .In general we take a_{n+1} such that $b_n = Sa_n = Ta_{n+1}$,n=0,1,2,...... from given (iii) in hypothesis,we have

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq \alpha G(Sa_n, Ta_{n+1}, Ta_{n+1}) + \beta G(Ta_n, Sa_{n+1}, Ta_{n+1}) + \gamma G(Ta_n, Ta_{n+1}, Sa_{n+1}) + \delta G(Sa_n, Ta_{n+1}, Ta_{n+1})$$

by construction of sequence, we have

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq \alpha G(Sa_n, Sa_n, Sa_n) + \beta G(Sa_{n-1}, Sa_{n+1}, Sa_n) + \gamma G(Sa_{n-1}, Sa_n, Sa_{n+1}) + \delta G(Sa_n, Sa_n, Sa_n)$$

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq \beta G(Sa_{n-1}, Sa_{n+1}, Sa_n) + \gamma G(Sa_{n-1}, Sa_n, Sa_{n+1})$$

since by symmetry in variables, we have.

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq (\beta + \gamma) G(Sa_{n-1}, Sa_n, Sa_{n+1}) \tag{3.1}$$

By using definition (2.2.1)(5), we have

$$G(Sa_{n-1}, Sa_n, Sa_{n+1}) \leq G(Sa_{n-1}, Sa_n, Sa_n) + G(Sa_n, Sa_n, Sa_{n+1}) \leq G(Sa_{n-1}, Sa_n, Sa_n) + 2G(Sa_n, Sa_{n+1}, Sa_{n+1})$$

(since by using proposition 2.2.2 (3)) from given inequality (iii) we have

$$(1 - 2\beta - 2\gamma) G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq (\beta + \gamma) G(Sa_{n-1}, Sa_n, Sa_n)$$

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq \frac{(\beta + \gamma)}{(1 - 2\beta - 2\gamma)} G(Sa_{n-1}, Sa_n, Sa_n)$$

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq qG(Sa_{n-1}, Sa_n, Sa_n)$$

where

$$q = \frac{(\beta + \gamma)}{(1 - 2\beta - 2\gamma)} < 1$$

continuing in this way, we have

$$G(Sa_n, Sa_{n+1}, Sa_{n+1}) \leq q^n G(Sa_0, Sa_1, Sa_1) \tag{3.2}$$

$\therefore$ for all n,m $\in N$ let $m > n$, By using rectangular inequality we have, Lets consider,

$$G(b_n, b_m, b_m) \leq G(b_n, b_{n+1}, b_{n+1}) + G(b_{n+1}, b_{n+2}, b_{n+2}) + ... + G(b_{m-1}, b_m, b_m)$$

$$G(b_n, b_m, b_m) \leq (q^n + q^{n+1} + ... + q^{m-1}) G(b_0, b_1, b_1) (since\ by\ using\ (3.4.3))$$

$$\leq \frac{q^n}{1-q} G(b_0, b_1, b_1)$$

as n,m $\to \infty$, since $q < 1$,$\therefore$ $\frac{q^n}{1-q} \to 0$ as n,m $\to \infty$ therefore R.H.S.of this inequality tends to 0. $\therefore$ we have $\lim_{n\to\infty} G(b_n, b_m, b_m) = 0$ Thus $\{b_n\}$ is a G-cauchy sequence in X. Also as X is a complete G-metric space, $\therefore$ there exists $c \in X$ s.t. $\{b_n\}$ G-converges to c. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} Sa_n = \lim_{n\to\infty} Ta_{n+1} = c$ Given S or T is continuous, Let T is continuous, $\lim_{n\to\infty} TSa_n = \lim_{n\to\infty} TTa_n = Tc$ Also S and T are compatible, $\therefore G(STa_n, TSa_n, TSa_n) = 0$ this gives $\lim_{n\to\infty} STa_n = Tc$ Now from hypothesis (iii) we have

$$G(STa_n, Sa_n, Sa_n) \leq \alpha G(STa_n, Ta_n, Ta_n) + \beta G(TTa_n, Sa_n, Ta_n) + \gamma G(TTa_n, Ta_n, Sa_n) + \delta G(STa_n, Ta_n, Ta_n)$$

taking lim $as\ n \to \infty$, we have Tc=c Again from (iii) we have

$$G(Sa_n, Sc, Sc) \leq \alpha G(Sa_n, Tc, Tc) + \beta G(Ta_n, Sc, Tc) + \gamma G(Ta_n, Tc, Sc) + \delta G(Sa_n, Tc, Tc)$$

Taking limit as $n \to \infty$, we have c=Sc. $\therefore$ we have Tc=Sc=c.This shows that c is a common fixed point of S and T.

uniqueness:-If possible c_1 other than c be another common fixed point of S and T.Then $G(c, c_1, c_1) > 0$ and

$$G(c, c_1, c_1) = G(Sc, Sc_1, Sc_1) \leq \alpha G(Sc, Tc_1, Tc_1) + \beta G(Tc, Sc_1, Tc_1) + \gamma G(Tc, Tc_1, Sc_1) + \delta G(Sc, Tc_1, Tc_1)$$

$G(c, c_1, c_1) \leq (\alpha + \beta + \gamma + \delta) G(c, c_1, c_1)$ $G(c, c_1, c_1) < G(c, c_1, c_1)$ □

References

[1] G.Jungck *Commuting maps and fixed points* Amer.Math.Monthly,83(1976)261-263.

[2] Sesa,S. *On Weakly Commutativity condition in fixed point consideration* Publ.Int.Math.32(46)(1986)

[3] K.M.Das,K.V.Naik, *Common fixed point theorems for commuting maps on a metric space* Proc.Amer.Math.Soc.,77(1979),369-373.

[4] B.Fisher *Common fixed points of four mappings* ,Bull.Inst.Acad.Sin.,11(1983),103-113.

[5] G.Jungck *Compatible mappings and common fixed points,* Internat.J.Math.Sci.9,(1986),771-779

[6] Jungck.G. , *Compatible mappings and common fixed points* (2),Int.J.Math.Math.Sci,11(1988),285-288

[7] G.Jungck *Common Fixed points for Commuting and Compatible maps on Compacta* Proc.Amer.Math.Soc.,103(1988),977-983

[8] Manoj Kumar *Compatible Maps in G-Metric spaces* Int.Journal of math. Analysis,Vol. 6,2012,no. 29,1415-1421.

[9] Mustafa Z.,Sims *A new approach to Generalized metric Spaces* J. nonlinear convex Anal. 7(2),289-297.(2006)

[10] Madhusudan Rao J. *Mappings On Metric Spaces* Indian J.Pure appl.Math.,11(4)(1980):502-506