ISBN No.978-93-82995-25-8

Advanced Studies in Pure Science and Applied Science

कला, वाणिज्य व विज्ञान
कनिष्ठ व वरिष्ठ महाविद्यालय

Chief Editor

Dr. B. M. Dhoot

Co-Editor

Dr. S.V. Kshirsagar

Dr. S.U. Kalme

Dr. P.R. Surve

Advanced Studies in Pure Science and Applied Science

Chief Editor
Dr. B. M. Dhoot

Co-Editor

Dr. S.V. Kshirsagar

Dr. S.U. Kalme

Dr. P.R. Surve

ISBN No. 978-93-82995-25-8

Published by:

Anuradha Publications

Cidco-Nanded

Publication Year: 2018-19

Price- Rs. 250/-

Printed by

Gurukrupa Offset,

Near Police Station, Gangakhed

Typesetting by:

Simran Computers

Gangakhed Dist.Parbhani

Cover Designby:

Mr. Imran K. Mohammad

CONTENTS

Sr. No. **Content**

*****

Fixed Point Theorem of Delbosco Contraction in Complete Metric Space

Mr. Latpate V.V.
ACS College Gangakhed,
Vishnu.latpate@yahoo.com

Abstract:

In this paper we define Delbosco contraction and Prove common fixed point theorem of Delboso contraction in Complete Metric Space.

AMS Subject classification:-47H10, 47H09

Keywords:-Cauchy Sequence, Complete Metric space, Fixed Point, Common fixed Point.

1. Introduction

S.Banach [1] proved Banach Contraction principle, which states that a contraction map on a complete Metric Space has a unique fixed point.In 1968 R.Kanan [2] proved a fixed point theorem for self map. In 1971 Chatterjea [3] proved a fixed point theorem for a self map which is a modification of Kanan map.

In 1980 Khan M.S. [4] proved some fixed point theorems in Metric and Banach Space. Afterward Delbosko [5] defined set of all continuous functions $g : R_+^3 \to R_+$ which satisfying some properties and proved some fixed point results.

2. Preliminaries

Definition 2.1:- Let X be a non-empty set . A mapping $d : X \times X \to R$ is said to be a Metric or a distance function if it satisfies following conditions.

1. $d(x,y)$ is non-negative.
2. $d(x,y)$=0 if and only if x and y coincides i.e. x = y.
3. $d(x,y) = d(y,x)$ (Symmetry)
4. $d(x,y) \le d(x,z) + d(z,y)$ (Triangle inequality)

Then the function d is referred to as metric on X. And (X,d) or simply X is said to as Metric space.

Definition 2.2:- A Metric space (X,d) is said to be a complete Metric space if every Cauchy sequence in X converges to a point of X.

Definition 2.3:- If (X,d) be a complete Metric space and a function $F : X \to X$ is said to be a contraction map if

$$d(F(x), F(y)) \le \beta d(x,y)$$

For all $x, y \in X$ and for $0 < \beta < 1$

Theorem 2.2 [3] A mapping $F : X \to X$ where (X,d) is a Metric space is said to be C-Contraction if there is a some β s.t. $0 < \beta < \frac{1}{2}$ s.t. the following inequality holds

$$d(F_x, F_y) \le \beta(d(x, F_y) + d(y, F_x))$$

If (X,d) be a complete Metric space, then any C-contraction on X has a unique fixed point.

In 1981 Delbosco [5] defined the set G of all continuous mappings α: $[0,\infty)^3 \to [0,\infty)$ which satisfies the following conditions

(i) $\alpha(1,1,1) = k < 1$

(ii) Let a,$b \ge 0$ be such that either a $\le k(a,b,b)$ or a $\le k(b,b,a)$ or a $\le k(b,a,b)$.Then a $\le kb$.

And Delbosco proved that for $P: X \to X, Q: X \to X$ on a Complete metric space (X,d) Which satisfies the condition,

$d(Pa,Qb) \le \alpha(d(a,b),d(a,Qb),d(b,Qb))$ (1)

For all $a,b \in X$,where α in G.Then P and Q have a unique common fixed point. Then in 1994 Adrian Constantin proved following result of common fixed point theorem.

Let P and Q be two self maps of a metric space (X,d) which satisfies following conditions

(i) $d(Pa,Qb) \le \alpha(d(a,b),d(a,Pa),d(b,Qb))$

for all a,b $\in$ X and $\alpha \in G$

(ii)There is a point v $\in X$ so that P is continuous at v and Q is continuous at Pv,

(iii)There exists a point a $\in X$ s.t. the sequence $\{(P \circ S)^n(a)\} = \{(Q \circ P)^n\}$ has a subsequence $\{(QP)^{n_i}(a)\}$ converging to v.Then v=Pu is the unique fixed point of P and Q.

And Delbosco proved common fixed point theorem for pair of two weakly commuting mappings also

Theorem 2.3 [5] If P and R be weakly commuting mappings and if Q and S be weakly commuting self mappings of a complete metric space (X,d) into itself which satisfies the following conditions

$d(Px,Qy) \le \alpha(d(Rx,Sy),d(Rx,Px),d(Sy,Qy))$ for all x,y òX

for αòG.If the range of R contains the range of Q and the range of S contains range of P, and if one of P,Q,R and S is continuous ,then P,Q,R and S have a unique common fixed point z.

Now we modify Delboscos inequality for two maps given by equation (1) and prove some common fixed point theorems .

First we introduce β Let E be the set of all functions $\beta:[0,\infty)^3 \to [0,\infty)$ which satisfies

(i) $\beta$is continuous on the set $[0,\infty)^3$ (with respect to Euclidean metric on $[0,\infty)^3$)

(ii) x $\le ky$ for some k s.t. $0 \le k < 1$ whenever x $\le \beta(y,y,x)$ or x $\le \beta(y,x,y)$ or x $\le \beta(x,y,y)$ for all x,y ò $[0,\infty)$

(iii) $\beta(x,y,y) = 0$ iff x=y=0

Definition 2.6 :- A mapping F on a metric space X into itself is said to be Delbosco contraction if it satisfies the following condition.

$d(Fa,Fb) \le \beta(d(a,b),d(a,Ta),d(b,Fb))$

for every a,b ò X and some βòE.

Example 2.1 :- A mapping $F: X \to X$ defined by

$d(Fa,Fb) \le \alpha \max\{d(Fa,a)+d(Fb,b), d(Fb,b)+d(a,b), d(Fa,a)+d(a,b)\}$

for all a,b in X and some $0 \le \alpha < \frac{1}{2}$ is New type contraction.

The map $\beta : R_+^3 \to R_+$ is defined as

$\beta(u,v,w) = \alpha \max\{u+v, v+w, u+w\}$

for every u,v,w ò R_+, *where* α is s.t. $0 \le \alpha < \frac{1}{2}$. Then as β òE.

Clearly β is continuous. Also for $u \le \beta(u,v,v) = \alpha \max\{u+v, v+u, v+v\}$

There are two possibilities

Case (i) If $\max\{u+v, v+u, v+v\} = u+v$

$\therefore$ In this case

$u \le \frac{\alpha}{1-\alpha} \le kv$, where k=$\frac{\alpha}{1-\alpha}$ in [0,1)

Case(ii) If

Case (ii) If $\max\{u+v, v+u, v+v\} = v+v$

$\therefore$ $u \le kv$ where k=2α, for $0 \le \alpha < 1$

$\therefore$ we have for $u \le \beta(v,u,v)$ or $u \le \beta(v,v,u)$ we obtain $u \le kv$ for k in $0 \le \alpha < 1$.

$$d(Fa,Fb) \le \alpha \max\{d(Fa,a)+d(Fb,b), d(Fb,b)+d(a,b), d(Fa,a)+d(a,b)\}$$
$$= \beta(d(a,b), d(Fa,a), d(Fb,b))$$

$\therefore$ by definition F is Delbosco contraction.

3 Main Result

Now we prove a fixed point result for Delbosco contraction.

Theorem 3.1 :- If (X;d) be a Complete Metric space and if F be mapping on X into itself which satisfies

$$d(Fa,Fb) \le \beta(d(a,b), d(a,Ta), d(b,Fb)) \quad (3.1)$$

for all a,b ò X and some βòX.

F has a unique fixed point in X.

Proof :- Let a_0 be an arbitrary point in X. We construct a sequence $\{a_n\}$ in X as

$a_{n+1} = Fa_n$, $a_1 = Fa_0$, $a_2 = Fa_1, \ldots\ldots, a_{n+1} = Fa_n$ i.e. $a_n = F^n a_0$

Given F satisfies (3.1)

Consider,

$$\begin{aligned} d(a_n,a_{n+1}) &= d(Fa_{n-1}, Fa_n) \\ &\le \beta(d(a_{n-1},a_n), d(Fa_{n-1},a_{n-1}), d(Fa_n,a_n)) \\ &\le \beta(d(a_{n-1},a_n), d(a_n,a_{n-1}), d(a_{n+1},a_n)) \\ &\le kd(a_{n-1},a_n) \qquad (3.2) \end{aligned}$$

Similarly,

$d(a_{n-1}, a_n) \le kd(a_{n-2}, a_{n-1})$

$\therefore$ (3.2) gives

$d(a_n, a_{n+1}) \le k^2 d(a_{n-2}, a_{n-1})$

. .

. .

. .

$$d(a_n, a_{n+1}) \le k^n d(a_0, a_1) \quad (3.3)$$

for $0 \le k < 1$

Letting $n \to \infty$ we have $\{a_n\}$ is a Cauchy sequence in X. And as X is complete.

$\therefore \{a_n\}$ converges to a point in X. Let $\{a_n\}$ converges to $u\grave{o}X$

$\therefore$ for a=u and b=a_n

inequality (3.1) gives,

d(Fa,a_{n+1}) $= d(Fu, Fa_n)$

$\le \beta(d(u,a_n), d(u,Fu), d(a_n, Fa_n))$

$= \beta(d(u,a_n), d(u,Fu), d(a_n, a_{n+1}))$

Taking limit as $n \to \infty$ and since given β is continuous we have

$d(Fu,u) \le \beta(d(u,u), d(u,Fu), d(u,u))$

$\therefore d(Fu,u) \le k.0 = 0$

Thus d(Fu,u)=0

this gives Fu=u

$\therefore u$ is a fixed point of F.

Uniqueness: - Now if possible suppose x be another fixed point of F

$\therefore Fx = x$

Now we put a=x and b=u in inequality (3.1), we have

Consider,

$d(x,u) = d(Tx,u)$

$\le \beta(d(x,u), d(Fx,x), d(Fu,u))$

$\le \beta(d(x,u), d(w,w), d(u,u))$

$\le \beta(d(x,u), 0, 0)$

$\therefore d(x,u) \le k.0$

$\therefore d(x,u) = 0$

This gives x=u.

References:-

[1] S.Banach. , Surles operations dansles ensembles abstraits et leur applications aux equations integrals, Fund. Math., 3(1922) 133-181.

[2] R. Kanan (1968) Some results on fixed points. Bull. Calcutta math. Soc, 60,182, 71-76

[3] S.K. Chatterjea, Fixed point theorems, C.R. Acad. Blgare Sci., 25(1972) 727-730.

[4] Khan M.S., on fixed point theorems. Math. Japonica 23(2) (1978/79), 201-204.

[5] Delbosko, D., A Unified approach for all contractive mappings, Inst. Math. Univ. Torino, report Nr. 19 (1981)

[6] Adrian Constantin, on some fixed point theorems in Metric spaces, Univ. u, Novom sadu zb. Rad prirod, Mat. Fak. Ser Mat. 24, 2(1994), 9-21.

******